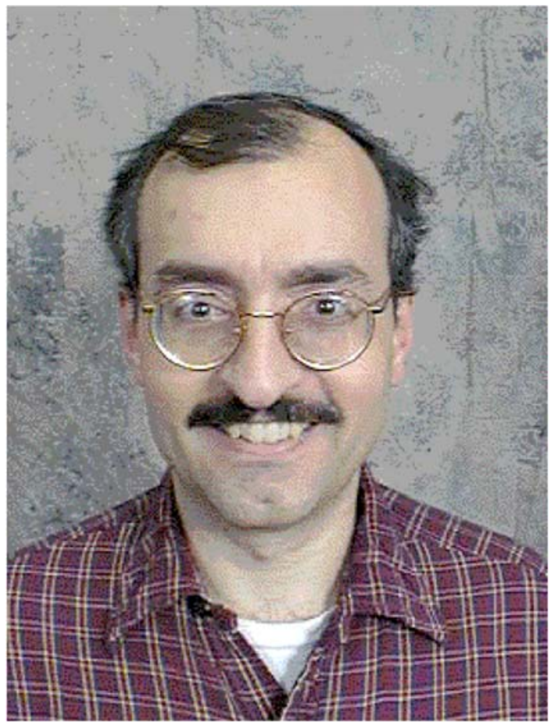
Bayesian Consensus?

Teddy Seidenfeld – Carnegie Mellon University

based on joint work with Jay Kadane and



Mark Schervish

Motivating the need to relax Subjective Expected Utility [SEU] theory in the direction of Indeterminate Probabilities.
The case of cooperative Bayesian decision making.

Consider two *SEU* Bayesian decision makers, Dick and Jane, who wish to form a cooperative partnership that will make decisions, constrained by the following two principles governing *coherence* and *compromise*.

- **The partnership, a group of two agents, itself a deciding agent, must satisfy the theory of Subjective Expected Utility maximization**
  The Group has a probability and a utility (P_G, U_G).
  The Group maximizes expected utility with respect to this pair.

- **(Simple) Pareto coordination – if each of Dick and Jane strictly prefers one option o_1 to a second o_2, then so too does the partnership.**

What are the candidate Bayesian compromises between Dick and Jane that may serve as the partnership's SEU preferences?

Background – ARROW'S "Impossibility" Theorem (1950)

A Difficulty in the Concept of Social Welfare, Kenneth J. Arrow *J. Political Economy*, Vol. **58**: 328-346.

Consider a (finite) set of *m*-many SOCIAL ACTS $A = \{ A_1, \ldots, A_m \}$, and

***n*-many INDIVIDUAL PREFERENCES over these social acts $\{ \leq_1, \ldots, \leq_n \}$.**

A *PREFERENCE*, $\leq$, is a *binary weak ordering* (*transitive* and *complete*) of the set A.

Arrow's Theorem:

> **There does not exist a rule for creating a GROUP PREFERENCE, $\leq_G$,**
> **that satisfies the following 4 conditions:**

(<u>C-1</u>) The rule applies with ARBITRARY sets of ACTS and PREFERENCES.

(<u>C-2</u>) The rule obeys the (WEAK) PARETO AXIOM:

$$\textbf{If } A_1 \leq_j A_2 \textbf{ (for each j), then } A_1 \leq_G A_2.$$

That is, when each person (weakly) prefers A_2 to A_1, the group does too.

(<u>C-3</u>) A DICTATOR is not permitted.

(<u>C-4</u>) The GROUP'S preference relation, $\leq_G$, over a particular subset *A′* of social acts,

e.g., $\leq_G$ applied to the odd-numbered social acts,

depends solely on the INDIVIDUALS' PREFERENCES, $\leq_j$, for the acts in *A′*.

Condition (C-4) is also called,

INDEPENDENCE OF IRRELEVANT ALTERNATIVES.

Possibility/Impossibility Results for Cooperative SEU compromises.

What are the candidate Bayesian compromises between Dick and Jane that may serve as the partnership's SEU preferences subject to Pareto?

Aside: We use the framework of Anscombe-Aumann horse lotteries, where there are no moral hazards and utility is state-independent.

1. **If Dick and Jane share a common cardinal utility over outcomes, the candidate compromises for the group's binary preferences are given by an average of their two personal probabilities, and the common utility.**

$$U_G = U_1 = U_2 \text{ and } P_G = \alpha P_1 + (1-\alpha)P_2 \quad (0 \leq \alpha \leq 1) \qquad \text{(Harsanyi)}$$

2. **If Dick and Jane share a common personal probability over the states, the candidate compromises for the group's binary preferences are given by an average of their two cardinal utilities, and the common probability. (Harsanyi, many others too.)**

$$P_G = P_1 = P_2 \text{ and } U_G = \alpha U_1 + (1-\alpha)U_2 \; (0 \leq \alpha \leq 1).$$

3. If Dick and Jane have any difference in their personal probability and do not share the same cardinal utility over rewards there are only autocratic solutions. (SSK – 1989, for the basic result.)

One of them makes *all* the decisions for the partnership!

$$P_G = P_1 \ \& \ U_G = U_1 \quad \text{or} \quad P_G = P_2 \ \& \ U_G = U_2.$$

Aside: The situation with more than two partners is complicated. Even with acts defined on a binary partition, with three Bayesian agents (so that one must be a linear combination of the other two) there can be cases with only autocratic Pareto solutions. (Jay Goodman, Ph.D. thesis, 1988.)

- **In short, marriage counseling is not as simple as the directive to enlarge the partnership by having children!**

Example 1: A Heuristic Illustration of the Problem and a Proof

Assume probabilities are act/state independent – no moral hazards.
Assume that utilities are state-independent – problems of Small Worlds.

Suppose Dick and Jane have different beliefs, $P_1 \neq P_2$.

Let $P_1(E) = .1$ and $P_2(E) = .3$ for some event E.

Also, suppose they have different values, $U_1 \neq U_2$.

Assume that each prefers reward r* to r_*, though they differ in their valuation of a third reward r.

$U_1(r) = .1$ & $U_2(r) = .4$, while $U_1(r_*) = U_2(r_*) = 0$ & $U_1(r^*) = U_2(r^*) = 1$.

Thus, Dick and Jane agree that

$$.1 \leq P_k(E) \leq .3 \qquad (k = 1, 2).$$

And they agree that

$$.1 \leq U_k(r) \leq .4 \qquad (k = 1, 2).$$

Table 1: "Horse lotteries" used in fixing the upper and lower probabilities and utilities.

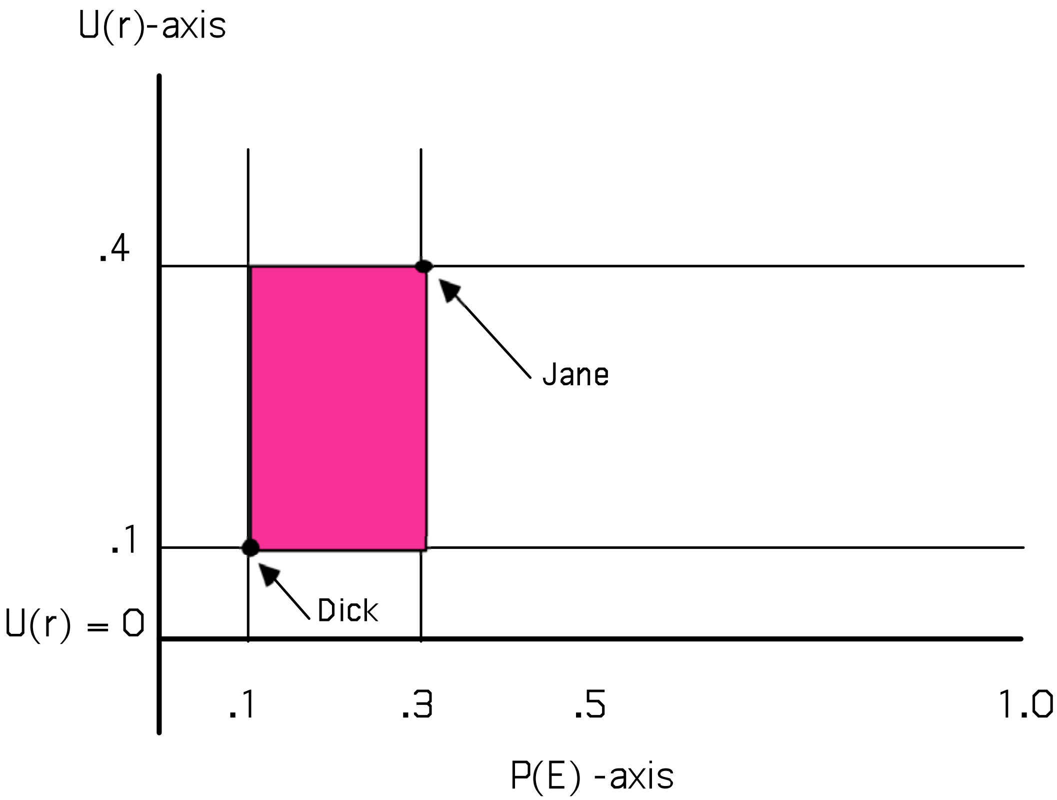
	E	$\neg E$
A_1	r^*	r_*
$A_{2\epsilon}$	$(.9+\epsilon)(r_*)+(.1-\epsilon)(r^*)$	$(.9+\epsilon)(r_*)+(.1-\epsilon)(r^*)$
$A_{3\epsilon}$	$(.7-\epsilon)(r_*)+(.3+\epsilon)(r^*)$	$(.7-\epsilon)(r_*)+(.3+\epsilon)(r^*)$
A_4	r	r
$A_{5\epsilon}$	$(.6-\epsilon)(r_*)+(.4+\epsilon)(r^*)$	$(.6-\epsilon)(r_*)+(.4+\epsilon)(r^*)$

The (weak) Pareto condition results in the following:

- **$A_{2\varepsilon} <_G A_1 <_G A_{3\varepsilon}$ fixing lower and upper probabilities for E.**
- **$A_{2\varepsilon} <_G A_4 <_G A_{5\varepsilon}$ fixing lower and upper utilities for r.**

These induce common strict preferences among gambles, whose implications for coherent extensions of $<_G$ are pictured in Figure 1.

Figure 1


U(r)-axis
.4
.1
U(r) = 0
Jane
Dick
.1
.3
.5
1.0
P(E) -axis


designates the set of probability/utility pairs
agreeing with the common preferences of Dick
and Jane for the comparisons, above.

However, also Dick and Jane share a common preference among pairs of acts of the following sort:

Table 2: "Horse lotteries" used in separating the set of compromises between Dick and Jane.

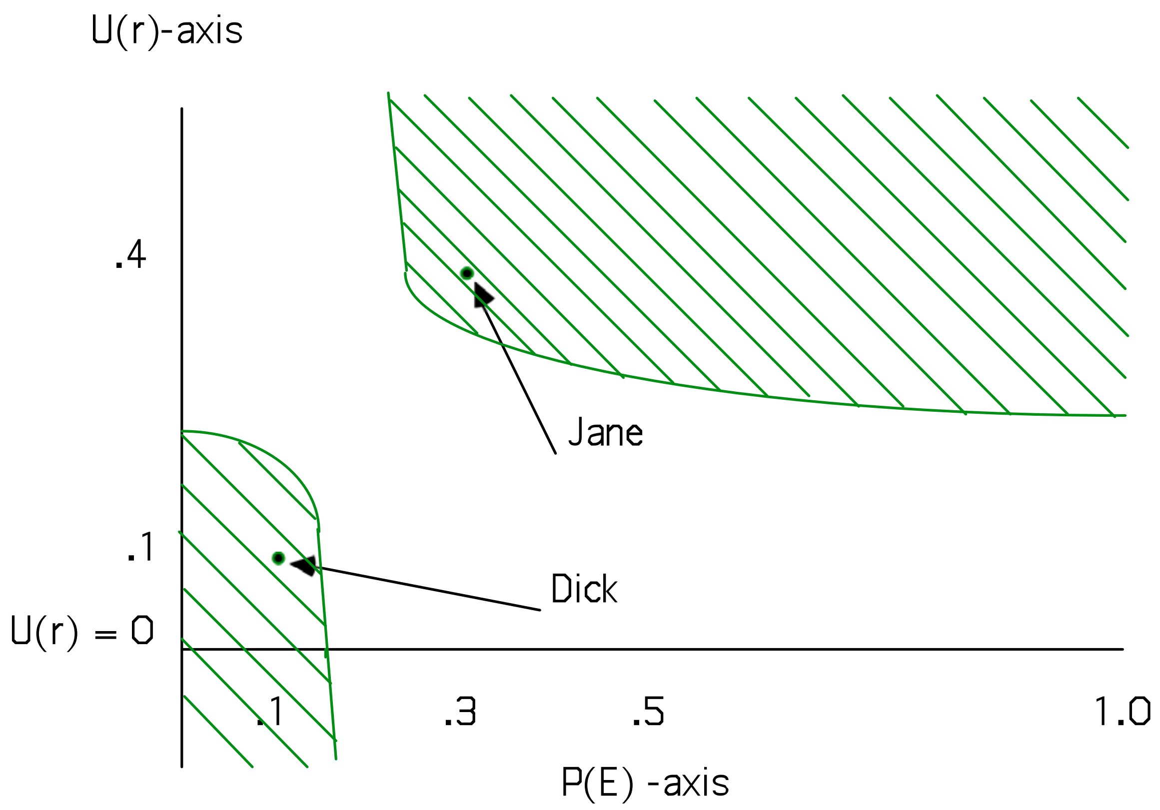
	E	$\neg E$
$A_{6\epsilon}$	$.785(r_*) + .215(r^*)$	$\epsilon(r_*) + .2(r) + (.8 - \epsilon)(r^*)$
$A_{7\epsilon}$	$(.2 - \epsilon)(r_*) + .8(r) + \epsilon(r^*)$	$.165(r_*) + .835(r^*)$

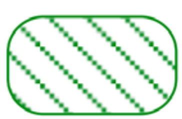
The (weak) Pareto condition also fixes these group preferences

- $\mathbf{A_{6\varepsilon} <_G A_{7\varepsilon}}$ **for** $\mathbf{0 < \varepsilon < 0.015}$**,**

which has the following geometric interpretation.

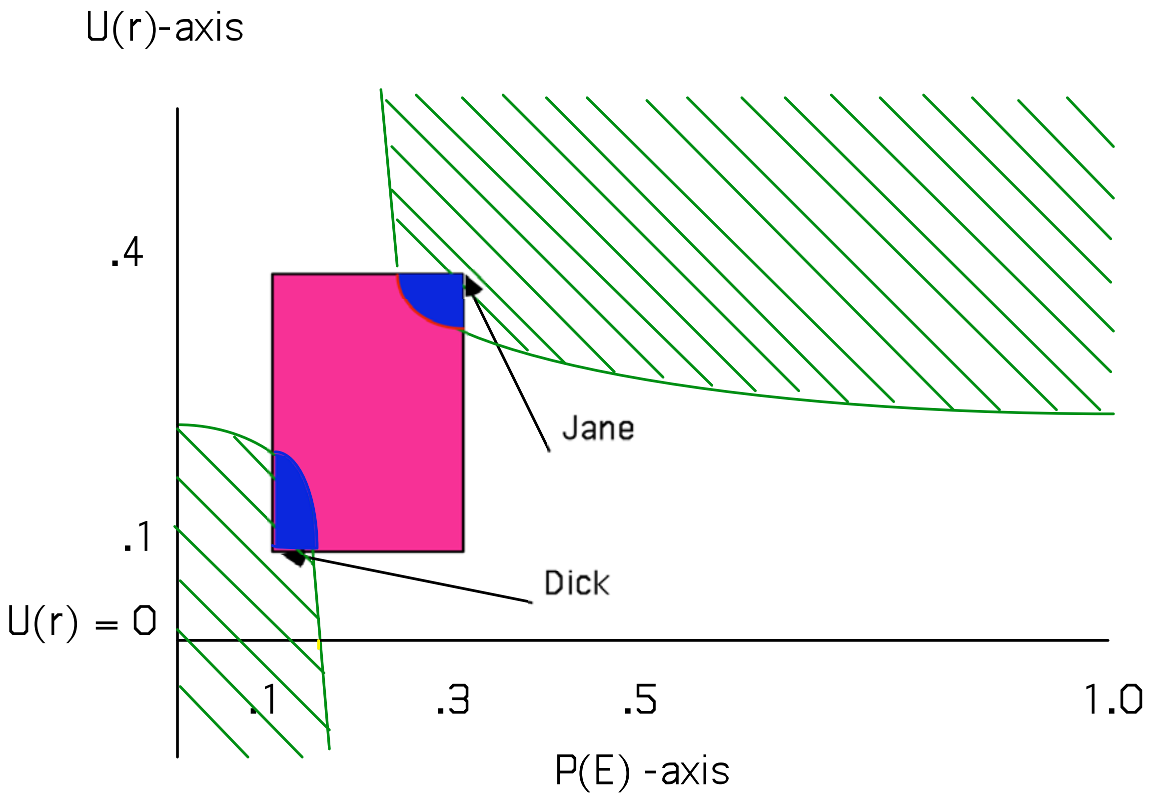
Figure of preferences which separate
the family of agreeing
probability/utility pairs.

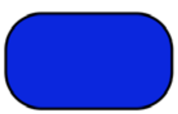



designates the set of probability/utility pairs agreeing with the common preferences of Dick and Jane for A7 over A6, e = .01.

If we superimpose the two figures, we obtain

Figure 3





designates the set of probability/utility pairs agreeing with the common preferences of Dick and Jane, $e = .01$.

Figure 3 shows that the set of coherent extensions of the partial order $<_G$ is disconnected .

The set of coherent extensions is not convex.

A pseudo-corollary for academic administrators:

Stop meetings as soon as there is unanimity on what needs to be done!

Do not allow the faculty to seek a shared rationale for unanimous decisions.

Often enough they will agree on what to do; but almost surely they will not agree on why it should be done!

Contrasts between this result and Arrow's "Impossiblity" for cooperative decision making.

1. **Arrow's result permits individuals and society to have any weak-order preference. However, we restrict all weak preference orders to Bayesian SEU rankings.**

2. **Arrow's condition C-4 (*IIA*) precludes interpersonal cardinal utility comparisons and without C-4 there are "possibility" results under conditions C-1, C-2, and C-3. However, we do not impose C-4.**

3. **Arrow's "Impossibility" reads,**

"For each consensus rule there will be a preference profile where …"

However, the result for the Bayesian Impossibility reads,

"For each pair of Bayesians with different values and beliefs there is no Bayesian consensus of SEU preferences that respects Pareto."

Combining Expert Bayesian Opinions. Can it be done?

Having seen that it is impossible to create fully Bayesian (non-autocratic) compromises that involve both beliefs and values, let us reconsider the simpler problem of merging different degrees of belief. The challenge is to determine whether there are defensible rules for combining a set of *n*-many "expert" probability distributions into one common probability distribution.

We suppose that each of our *n*-many experts has an opinion about some common domain of interest, represented by the partition into relevant states:

$$\Omega = \{\omega_1, \ldots, \omega_k\}.$$

Expert$_i$'s opinion is probability distribution $P_i = \langle p_{i1}, \ldots, p_{ik} \rangle$ over Ω, i =1, …, *n*.

- **Can we combine these *n*-many probabilities, P_i, into a single probability P_G that reflects the group's combined wisdom?**

<u>Linear Pooling</u>:

Assign each expert a non-negative *weight* $w_i \geq 0$ to reflect her/his relative expertise in the group, and standardize these so that $\sum_i w_i = 1$

Form $P_G = \sum_i w_i P_i$, the w_i-weighted average of their separate opinions.

- **P_G is called a *Linear Pool* of the expert opinions.**
- **The Linear Pool puts P_G inside the *hull* (= closed, convex set) of the n-many points P_i (i = 1, …, n).**

What are some of the nice features of a Linear Pool?

- **Preservation of unanimity of (unconditional) probabilistic opinions**

$$\text{If } c_1 \leq P_i(E) \leq c_2 \ (i = 1, \ldots, n) \text{ then } c_1 \leq P_G(E) \leq c_2.$$

Suppose there is a common utility U for outcomes across the group,
that is, suppose the group is a *Team*.

- **If each expert judges that Act_1 is better than Act_2 by the standards of SEU, then so too the Team will make the same Pareto judgment – using the shared utility U and pooled opinion P_G.**

- **The Linear Pool is computationally convenient in the following sense of being a *local* computation.**

Once the w_i ($i = 1, \ldots, n$) are fixed $P_G(E)$ depends solely on the *n*-values $P_i(E)$.

In other words, $P_G(E)$ does not depend upon how the n-many experts divide up their probabilities on E^c.

- **But is there a problem with the Linear Pool? What more might we want of a Bayesian consensus than is required by the Pareto condition for pairwise comparisons?**

Learning and Pooling.

Let us use conditional probability as the rule for updating new information.

- **$P_i(\bullet \mid F)$ is the revised opinion for P_i when new information F is added.**

(1) Consider allowing the experts all to learn the same new information F before pooling their opinions with weights w_i.

So, by this method of first updating and then pooling we obtain

$$P^1_G(\bullet \mid F) = \sum_i w_i P_i(\bullet \mid F).$$

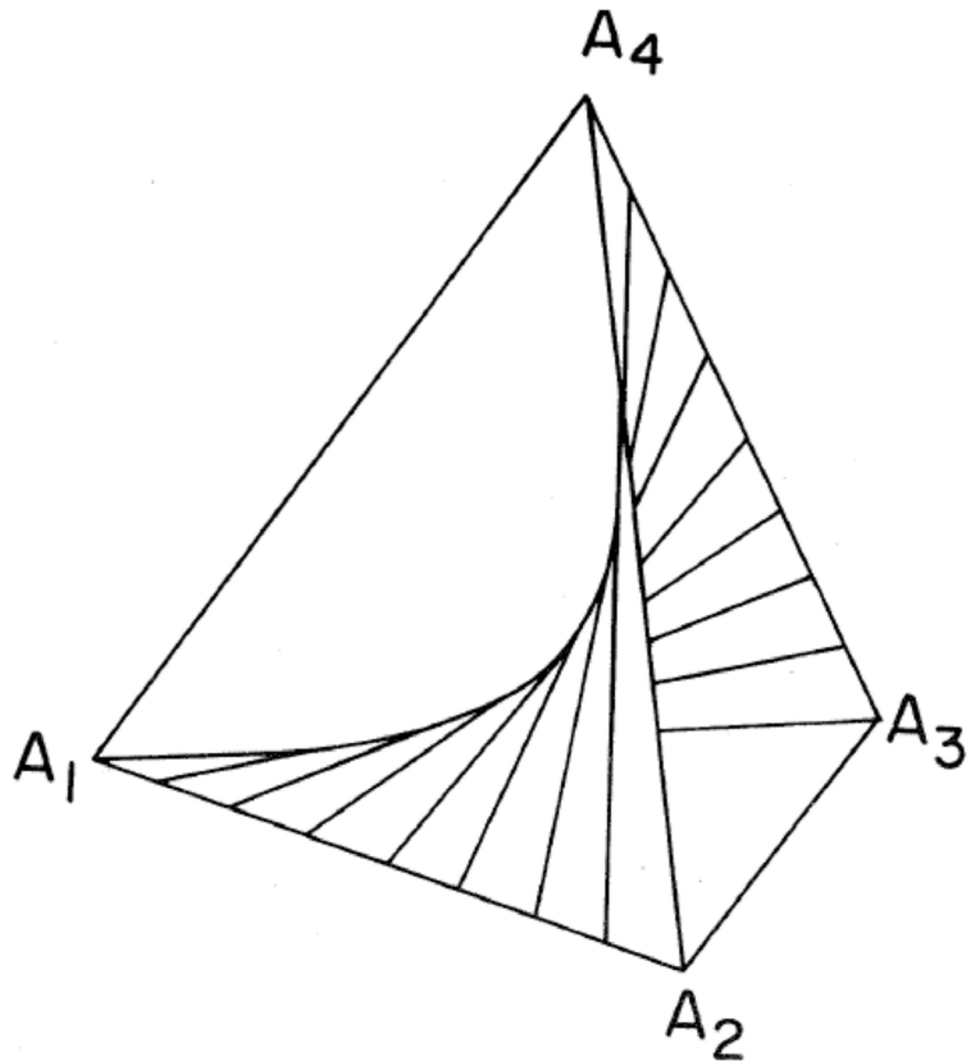
(2) However, we might first pool the expert opinions and then update P_G with the same information F, to yield

$$P^2_G(\bullet \mid F) = P_G(\bullet \cap F) \div P_G(F)$$
$$= \sum_i w_i P_i(\bullet \cap F) \div \sum_i w_i P_i(F).$$

Alas, generally, $P^1_G(\bullet \mid F) \neq P^2_G(\bullet \mid F)$ **!!**

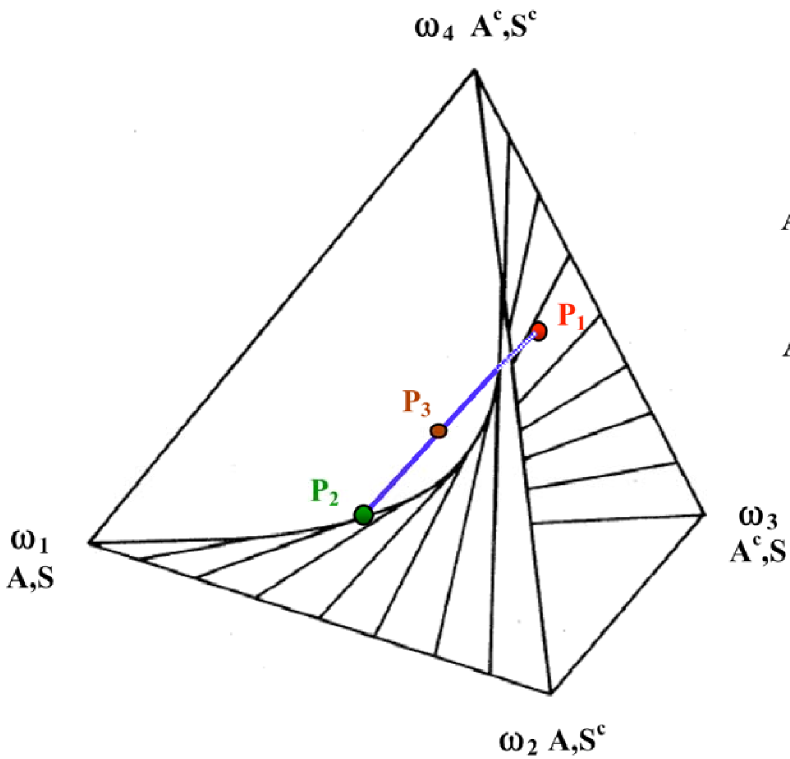
The Linear Pool is not "Externally Bayesian"!

Consider the 3-dimensional simplex of probabilities on two events.



	F	F^c
E	A_1	A_2
E^c	A_3	A_4

We see that, generally, linear pooling two probability distributions that make the events E and F independent will make them dependent!



	S	S^c
A	ω_1	ω_2
A^c	ω_3	ω_4

This method of pooling creates some strange decisions for the group.

If n = 2 and both experts think that E and F are independent events, then each will refuse to pay anything to learn about F before betting on E. However, if a linear opinion pool is formed first, that opinion may make E and F dependent events, and under the pooled-opinion, there will be value in first learning F before wagering on E.

Example 2 (sketch): Consider two doctors who are unsure both about your allergic state and about the weather in China, but who agree these are independent events. Do you mind if, instead of checking your medical history for information about your drug allergies, instead they spend the insurance money learning about the weather in China and using that information to decide on your treatment?

Here is the normal form version of that sequential problem.

***Example 2*: Consider a decision problem among three options – three treatment plans $\{T_1, T_2, T_3\}$ defined over 4 states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with determinate utility outcomes given in the following table. That is, the numbers in the table are the utility outcomes for the options (rows) in the respective states (columns).**

	ω_1	ω_2	ω_3	ω_4
T_1	0.00	0.00	1.00	1.00
T_2	1.00	1.00	0.00	0.00
T_3	0.99	-0.01	-0.01	0.99

Consider the convex set of probabilities be generated by two extreme points, distributions P_1 and P_2. Distribution P_3 is the .50-.50 (convex) mixture of P_1 and P_2.

	ω_1	ω_2	ω_3	ω_4
P_1	0.08	0.32	0.12	0.48
P_2	0.48	0.12	0.32	0.08
P_3	0.28	0.22	0.22	0.28

- **Note well that (for i = 1, 2, 3) under probability P_i, only option T_i is Bayes-admissible from the option set of $\{T_1, T_2, T_3\}$.**

Now, interpret these states as the cross product of two binary partitions:

- **a binary medical event – A (patient allegric), A^c (patient not-allergic),**
- **a binary meteorological partition – S (sunny) and S^c (cloudy).**

Specifically: $\omega_1 = A\&S \quad \omega_2 = A\&S^c \quad \omega_3 = A^c\&S \quad \omega_4 = A^c\&S^c$

- **Under P_1, the two partitions are independent events with $P_1(A) = .4$ and $P_1(S) = .2$.**
- **Likewise, under P_2, the events are independent, $P_2(A) = .6$ and $P_2(S) = .8$.**
- **But under linear pooling P_3, A and S are positively correlated:**

$$.56 = P_3(A \mid S) > P_3(A) = .5,$$

as happens with each distribution that is a non-trivial mixture of P_1 and P_2.

The three options have the following interpretations:

T_1 and T_2 are ordinary medical options, with outcomes that depend solely upon the patient's allergic state.

T_3 is an option that makes the allocation of medical treatment a function of the meteorological state, with a "fee" of 0.01 utile assessed for that input.

T_3 is the option "T_1 if cloudy and T_2 if sunny, while paying a fee of 0.01."

Suppose P_1 represents the opinion of medical expert$_1$, and P_2 represents the opinion of medical expert$_2$.

Without linear pooling, T_3 is inadmissible for each expert.

> **This captures the shared agreement between the two medical experts that T_3 is unacceptable from the choice of three $\{T_1, T_2, T_3\}$, and it captures the pre-systematic understanding that under T_3 you pay to use medically irrelevant inputs about the weather in order to determine the medical treatment.**

However, with linear pooling of the pair P_1 and P_2, then T_3 (or a variant of T_3) becomes uniquely admissible for the group.

- **There is no violation of the (binary) Pareto condition under the group opinion formed by the linear pool since the experts disagree about which option, T_1 or T_2, is better than T_3, though they agree that T_3 is not best.**

***Aside*: *Consensus* is not bargaining!**

From a *bargaining* point-of-view, it makes good sense for each expert to accept option T_3.

Option T_3 allows each party in a bargaining problem to think that, with probability .8, his/her medical view will decide the treatment allocation for the patient.

Externally Bayesian Pooling Rules.

There is a family of pooling rules that is invariant over the order of pooling and updating by conditioning. These are called *Externally Bayesian* Pooling rule.

It is a *Logarithmic Pool*: $\mathbf{P}_G \propto \prod_i \mathbf{P}_i^{w_i}$

It is a linear pool in the logarithms of the expert opinions.

- **What is problematic about this pooling rule?**

Example 3: Using three states and two experts.

$\Omega = \{\omega_1, \omega_2, \omega_3\}$ $\mathbf{P}_1 = <.3, .5, .2>$, $\mathbf{P}_2 = <.3, .2, .5>$, **and** $w_1 = w_2$.

***Exercise*: Show that using the logarithmic pooling rule,** $\mathbf{P}_G(\omega_1) \neq .3$**, which is a violation of unanimity for pooling of the unconditional probabilities.**

Summary

1. We require that a cooperative partnership obey the Pareto rule:
Preserve all strict preferences about which there is unanimity.

But there are no non-autocratic Bayesian compromises between two Bayesian agents who differ in their values (utilities) and in their beliefs (probabilities).

Aside: We use the framework of Anscombe-Aumann horse lotteries, where there are no moral hazards and utility is state-independent.

2. Even the more modest goal of finding consensus solely with respect to a set of probabilities is frustrated.

2.1 The Linear Pool is not Externally Bayesian and does not always preserve unanimous judgments about the value of information in sequential decisions.

2.2 The Logarithmic Pool is Externally Bayesian, but does not always preserve unanimity in unconditional probabilities.

Q*: *How to relax the SEU theory in order to avoid such results?